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FACTORIZATION THEOREMS OF CESÀRO AND COPSON SPACES ON TIME SCALES

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Abstract. In this paper, we prove some factorization theorems of Cesàro and Copson spaces on an arbitrary time scale $\mathbb{T}$, which offer enhancements of dynamic Copson's and Hardy's inequalities. Our results enhance, among others, the best-known forms of dynamic Hardy's inequality. The main results will be proved by employing the time scales Hölder's inequality and the derived time scales power rule of integration.

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1. Introduction

In 1920 Hardy [7] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p, \quad p>1. \tag{1.1}$$

This inequality has been discovered in his attempt to give an elementary proof of Hilbert's inequality for double series that was known at that time, where $a_n \geq 0$ for $n \geq 1$. In 1925 Hardy [8] proved the continuous inequality, using the calculus of variations, which states that for $f \geq 0$, integrable over any finite interval $(0, x)$, f^p is integrable, convergent over $(0, \infty)$ and $p > 1$, then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f^p(x)dx. \tag{1.2}$$

The constant $(p/(p-1))^p$ in (1.1) and (1.2) is the best possible. Hardy's inequalities (1.1) and (1.2), can be interpreted as inclusions between the space of sequences l_p (i.e. space of all sequences $(a_n)_{n\geq 1}$ such that $(\sum_{n=1}^{\infty}|a_k|^p)^{\frac{1}{p}} < \infty$) and Cesàro space of sequences (respectively functions). That is

$$l_p \subseteq \text{ces}(p),\ p>1.$$

The Cesàro space of sequences is defined to be the set of all real sequences $(a_n)_{n\geq 1}$ that satisfy

$$\|a\|_{Ces_p} = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|a_k|\right)^p\right)^{\frac{1}{p}} < \infty,$$

and the Cesàro space of functions is defined to be the set of all Lebesgue measurable real functions on $[0,\infty)$ such that

$$\|f\|_{Ces_p} = \left(\int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{\frac{1}{p}} < \infty.$$

The same interpretation is valid if the Hardy operator is substituted by its dual. In his celebrated book, Bennett [4, Theorem 4.5] "enhanced" the classical Hardy inequality by substituting it with an equality, factorizing the Cesàro space of sequences, with the final aim to characterize its Köthe dual. He proved that a sequence x belongs to the Cesàro space of sequences $\mathrm{ces}(p)$ if and only if it admits a factorization

$$x = y \cdot z \tag{1.3}$$

with

$$y \in l_p \text{ and } z_1^{p^*} + z_2^{p^*} + \cdots + z_n^{p^*} = O(n), \tag{1.4}$$

where $p^* = \frac{p}{p-1}$ is the conjugate index of p. This factorization gives also a better insight in the structure of Cesàro spaces. Since the discovery of this new way of looking at inequalities, various papers which deal with new proofs, generalizations and extensions have appeared in the literature. Several mathematicians such as Barza [3], Carton and Heinig (see [6]), Manna [12], Johnson and Mohapatra (see [9], [10] and [11]) studied the generalizations of the sequence spaces l_p and $\mathrm{ces}(p)$.

The natural question emerges now: Is it possible to extend the factorization concept (1.3) and (1.4) to an arbitrary time scale $\mathbb{T}$ and obtain their continuous and discrete analogues as special cases?. The aim of this paper is to give an affirmative answer to this question. In particular, we will prove the time scales version of Bennett's result. The main results will be proved by applying the time scales Hölder's inequality and the time scales power rule of integration.

The paper is organized in the following way: In Section 2, we give some basic concepts of the calculus on time scales and some other lemmas which will be used throughout the paper. In Section 3, we prove the main results of the paper.

2. Preliminaries and basic lemmas

In this section, we present some basic definitions concerning the delta calculus on time scales. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$. The mapping $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, \infty)$ such that $\mu(t) := \sigma(t) - t$ is called graininess. A function $f : [a, b] \to \mathbb{R}$ is said to be right–dense continuous ($rd-$continuous) if it is right continuous at each right–dense point and there exists a finite left limit at all left–dense points, and f is said to be differentiable if its derivative exists. The space of $rd-$continuous functions is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. In addition, we presume that $\sup \mathbb{T} = \infty$ and the time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$. For a function $f : \mathbb{T} \to \mathbb{R}$, we define the delta derivative f^{Δ} as follows: For $t \in \mathbb{T}$, if there exists a number $\alpha \in \mathbb{R}$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|,$$

for all $s \in U$, then f is said to be differentiable at t, and we call α the delta derivative of f at t denoted by $f^{\Delta}(t)$. For example, if $\mathbb{T} = \mathbb{R}$, then

$$f^{\Delta}(t) = f^{'}(t) = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \text{ for all } \ t \in \mathbb{T}.$$

If $\mathbb{T} = \mathbb{N}$, then $f^{\Delta}(t) = f(t+1) - f(t)$ for all $t \in \mathbb{T}$. A useful formula is $f^{\sigma} = f + \mu f^{\Delta}$, where $f^{\sigma} := f \circ \sigma$. The following theorem gives the product rule for the derivative of the product fg of two differentiable functions f and g. Assume f, $g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}$, Then

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}. \tag{2.1}$$

Now, we pass to the antiderivative and the integration on time scales for detla differentiable functions. For $a, b \in \mathbb{T}$, and a delta differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t) \Delta t = f(b) - f(a).$$

An integration by parts formula reads

$$\int_a^b f(t) g^{\Delta}(t) \Delta t = f(t) g(t)|_a^b - \int_a^b f^{\Delta}(t) g^{\sigma}(t) \Delta t. \tag{2.2}$$

More details about delta calculus on time scales and the corresponding integral can ce found in [5, Chapter 1]. In the following, we present a time scales chain rule. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f^{'}(g(t) + h\mu(t) g^{\Delta}(t)) dh \right\} g^{\Delta}(t), \tag{2.3}$$

holds. A special case of (2.3) is given by

$$(x^{\gamma}(t))^{\Delta}=\gamma\int_0^1\left[hx^{\sigma}+(1-h)x\right]^{\gamma-1}dhx^{\Delta}(t),\quad \gamma>0. \tag{2.4}$$

The Hölder inequality, see [5, Theorem 6.13], on time scales is given by

$$\int_a^b |f(t)g(t)|\Delta t\le\left[\int_a^b |f(t)|^{\gamma}\Delta t\right]^{\frac{1}{\gamma}}\left[\int_a^b |g(t)|^{\nu}\Delta t\right]^{\frac{1}{\nu}}, \tag{2.5}$$

where $a,\ b\in\mathbb{T}$ and $f,\ g\in \mathrm{C}_{rd}(\mathbb{I},\mathbb{R})$, $\gamma>1$ and $\frac{1}{\gamma}+\frac{1}{\nu}=1$.

In the following, we will introduce the time scales power rule for integration presented and proved in [14].

Lemma 2.1. *Let $\mathbb{T}$ be a time scale with $a,x\in\mathbb{T}$ and $x\ge a$. If $0<p<1$, then*

$$\left(\int_a^{\sigma(x)} f(t)\Delta t\right)^p\ge p\int_a^{\sigma(x)} f(t)\left(\int_a^{\sigma(t)} f(s)\Delta s\right)^{p-1}\Delta t, \tag{2.6}$$

the inequality reversed for $p\ge1$.

Lemma 2.2. *Let $\mathbb{T}$ be a time scale with $b\in\mathbb{T}$. If $0<p<1$, then*

$$\left(\int_x^b f(t)\Delta t\right)^p\ge p\int_x^b f(t)\left(\int_t^b f(s)\Delta s\right)^{p-1}\Delta t, \tag{2.7}$$

for $x\in\mathbb{T}$, $x\le b$. The inequality reversed for $p\ge1$.

The following special form of the dynamic Mikowski inequality, presented in [14],will be needed in the sequel.

Lemma 2.3. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and let f and g be nonnegative $rd-$continuous functions on $[a,b]_{\mathbb{T}}$. If $m\ge1$, then*

$$\left(\int_a^b f(t)\left(\int_a^{\sigma(t)} g(s)\Delta s\right)^m\Delta t\right)^{\frac{1}{m}}\le\int_a^b g(s)\left(\int_s^b f(t)\Delta t\right)^m\Delta s. \tag{2.8}$$

Finally, the following two Hardy's lemmas will play a remarkable role in the proofs of our main results, see [15, Page 476] for their detailed proofs.

Lemma 2.4. *Let $\mathbb{T}$ be a time scale with $a,\ b\in\mathbb{T}$ and $f,\ g,\ h\in \mathrm{C}_{rd}([a,b]_{\mathbb{T}},\mathbb{R}^{+})$. If*

$$\int_a^{\sigma(t)} f(x)\Delta x\le\int_a^{\sigma(t)} g(x)\Delta x,$$

then

$$\int_a^b f(t)H(t)\Delta t\le\int_a^b g(t)H(t)\Delta t, \tag{2.9}$$

where $H(t):=\int_t^b h(x)\Delta x$.

Lemma 2.5. *Let $\mathbb{T}$ be a time scale with a, $b \in \mathbb{T}$ and f, g, $h \in \mathrm{C}_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}^{+})$. If k is a positive constant such that*

$$F(t) := \int_t^b f(x)\Delta x \leq k \int_t^b g(x)\Delta x := kG(t),$$

then

$$\int_t^b f(t)H^{\sigma}(t)\Delta t \leq k \int_t^b g(t)H^{\sigma}(t)\Delta t, \tag{2.10}$$

where $H^{\sigma}(t) := \int_a^{\sigma(t)} h(x)\Delta x$.

3. Factorization theorems of Cesàro space

Throughout this section (without mentioning) the integrals in the statements of the theorems are assumed to exist. We assume throughout the paper that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$ and assume that the functions in the statements of the theorems are rd-continuous and $\Delta-$integrable functions defined on $[0,\infty)_{\mathbb{T}}$. We say that the function $f : [0,\infty)_{\mathbb{T}} \rightarrow \mathbb{R}$ belongs to the space $L^p_{\Delta}(\mathbb{T})$ if

$$\|f\|_{L^p_{\Delta}(\mathbb{T})} = \left(\int_0^{\infty} |f(t)|^p \Delta t\right)^{\frac{1}{p}} < \infty, \text{ if } 1 \leq p < \infty$$

or there exists a constant $C \in \mathbb{R}^{+}$ such that $\|f\|_{\infty} = \sup_{t\geq 0} |f(t)| < C$, if $p = +\infty$. The Cesàro space $Ces^p_{\Delta}(\mathbb{T})$ for $p \geq 1$ is the space of all functions f defined on $[0,\infty)_{\mathbb{T}}$ such that

$$\|f\|_{Ces^p_{\Delta}(\mathbb{T})} = \left(\int_0^{\infty} \left(\frac{1}{\sigma(x)} \int_0^{\sigma(x)} |f(t)|\, \Delta t\right)^p \Delta x\right)^{1/p} < \infty. \tag{3.1}$$

For $p \geq 1$, the space $Ces^p_{\lambda}(\mathbb{T})$ is obvious a Banach space with the norm (3.1). For $1 \leq p < \infty$, we define the function space $G_{\Delta}(p)$ as

$$G_{\Delta}(p) := \left\{ h : \sup_{x>0} \left(\frac{1}{\sigma(x)} \int\limits_0^{\sigma(x)} |h(t)|^p \,\Delta t \right)^{\frac{1}{p}} < \infty \right\},$$

and we denote

$$!f!_p = \inf\left\{ \|g\|_p \, \|h\|_{G_{\Delta}(p^*)} \right\},$$

where infimum is taken over all factorizations $f = g \cdot h$ with $g \in L^p_{\Delta}(\mathbb{T})$ and $h \in G_{\Delta}(p^*)$.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. If $1 < p < \infty$, then*

$$Ces^p_{\Delta}(\mathbb{T}) = L^p_{\Delta}(\mathbb{T}) \cdot G_{\Delta}(p^*), \tag{3.2}$$

which means that the function f belongs to $Ces_{\Delta}^{p}(\mathbb{T})$ if and only if it admits a factorization $f = g \cdot h$ with $g \in L_{\Delta}^{p}(\mathbb{T})$, $h \in G_{\Delta}(p^*)$ and

$$(p-1)^{\frac{-1}{p}} !f!_p \leq \|f\|_{Ces_{\Delta}^{p}} \leq p^* !f!_p. \tag{3.3}$$

Proof. "Imbedding $\hookrightarrow$". For $f \in Ces_{\Delta}^{p}(\mathbb{T})$, $f \neq 0$ and $x > 0$, let

$$k(x) = \int_x^\infty u^{-1} \left(\frac{1}{\sigma(u)} \int_0^{\sigma(u)} |f(t)| \, \Delta t \right)^{p-1} \Delta u.$$

Then $k(x) > 0$, k is decreasing and by Hölder's inequality (2.5)

$$\begin{aligned} k(x) &= \int_x^\infty u^{-1} \left(\frac{1}{\sigma(u)} \int_0^{\sigma(u)} |f(t)| \, \Delta t \right)^{p-1} \Delta u \\ &\leq \left(\int_x^\infty u^{-p} \Delta u \right)^{\frac{1}{p}} \left(\int_x^\infty \left(\frac{1}{\sigma(u)} \int_0^{\sigma(u)} |f(t)| \, \Delta t \right)^{p} \Delta u \right)^{\frac{1}{p^*}}. \end{aligned}$$

Using time scales chain rule (2.4) on the term $\left(\int_x^\infty u^{-p} \Delta u \right)^{\frac{1}{p}}$, we get that

$$\left(\int_x^\infty u^{-p} \Delta u \right)^{\frac{1}{p}} \leq \frac{1}{(p-1)^{\frac{1}{p}} x^{1-\frac{1}{p}}},$$

which leads directly to

$$k(x) \leq \frac{1}{(p-1)^{\frac{1}{p}} x^{1-\frac{1}{p}}} \|f\|_{Ces_{\Delta}^{p}}^{p-1}.$$

Consider the factorization $f = g \cdot h$, where

$$g(x) = (|f(x)| \, k(x))^{\frac{1}{p}} \ \operatorname{sgn} f(x) \qquad \text{and} \qquad h(x) = |f(x)|^{\frac{1}{p^*}} k(x)^{\frac{-1}{p}}.$$

Applying Lemma 2.3, we obtain that

$$\begin{aligned} \|g\|_p^p &= \int_0^\infty |f(x)| \int_x^\infty u^{-p} \left(\int_0^{\sigma(u)} |f(t)| \, \Delta t \right)^{p-1} \Delta u \Delta x \\ &\leq \int_0^\infty u^{-p} \left(\int_0^{\sigma(u)} |f(t)| \, \Delta t \right)^{p-1} \int_0^{\sigma(u)} |f(x)| \, \Delta x \Delta u = \|f\|_{Ces_{\Delta}^{p}}^{p}. \end{aligned}$$

By using Hölder's inequality (2.5), we get that

$$\begin{aligned} \left(\int_0^{\sigma(x)} |h(t)|^{p^*} \Delta t \right)^{p} &= \left(\int_0^{\sigma(x)} |f(t)|^{\frac{1}{p^*}} |f(t)|^{\frac{1}{p}} k(t)^{\frac{-p^*}{p}} \Delta t \right)^{p} \\ &\leq \left(\int_0^{\sigma(x)} |f(t)| \, \Delta t \right)^{p-1} \left(\int_0^{\sigma(x)} |f(t)| \, k(t)^{-p^*} \Delta t \right). \end{aligned}$$

Since $k(x)$ is decreasing, we have that

$$\begin{aligned}
&\int_x^\infty \left(\frac{1}{\sigma(s)}\int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t\right)^p \Delta s \\
\leq\ & \int_x^\infty s^{-1}\left[\left(\frac{1}{\sigma(s)}\int_0^{\sigma(x)} |f(t)|\,\Delta t\right)^{p-1}\left(\int_0^{\sigma(x)} |f(t)|\,k(t)^{-p^*}\Delta t\right)\right]\Delta s \\
=\ & k(x)\int_0^{\sigma(x)} |f(t)|\,k(t)^{-p^*}\Delta t \\
\leq\ & \int_0^{\sigma(x)} |f(t)|\,k(t)^{1-p^*}\Delta t = \int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t,
\end{aligned}$$

equivalently,

$$\int_x^\infty (\sigma(s))^{-p}\Delta s\left(\int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t\right)^{p-1} \leq 1,$$

which leads to

$$\left(\int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t\right)^{p-1} \leq (p-1)\,x^{p-1},$$

and hence (note that $x \leq \sigma(x)$)

$$\sup_{x>0}\frac{1}{\sigma(x)}\int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t \leq (p-1)^{\frac{1}{p-1}},$$

or $\|h\|_{G_\Delta(p^*)} \leq (p-1)^{\frac{1}{p}}$, which proved that

$$Ces_\Delta^p \subset L_\Delta^p(\mathbb{T})\cdot G_\Delta(p^*),$$

and $(p-1)^{\frac{-1}{p}}!f! \leq \|f\|_{Ces_\Delta^p}$ which proves the right hand side.

"Imbedding $\hookleftarrow$". Let $f = g\cdot h$ with $g \in L_\Delta^p(\mathbb{T})$, $h \in G_\Delta(p^*)$. Then

$$\int_0^{\sigma(x)} |h(t)|^{p^*}\Delta t \leq \|h\|_{G_\Delta(p^*)}^{p^*}\int_0^{\sigma(x)}\Delta t.$$

Applying Lemma 2.4 for any decreasing function w on $(0,\infty)_{\mathbb{T}}$, we get that

$$\int_0^{\sigma(x)} |h(t)|^{p^*} w(t)\Delta t \leq \|h\|_{G_\Delta(p^*)}^{p^*}\int_0^{\sigma(x)} w(t)\Delta t.$$

By Hölder's inequality (2.5), we find that

$$\begin{aligned}
\left(\int_0^{\sigma(x)} |f(t)|\,\Delta t\right)^p &= \left(\int_0^{\sigma(x)} |g(t)|\,w^{\frac{-1}{p^*}}(t)\,|h(t)|\,w^{\frac{1}{p^*}}(t)\Delta t\right)^p \\
&\leq \int_0^{\sigma(x)} |g(t)|^p\,w^{1-p}(t)\Delta t\left(\int_0^{\sigma(x)} |h(t)|^{p^*} w(t)\Delta t\right)^{p-1} \\
&\leq \int_0^{\sigma(x)} |g(t)|^p\,w^{1-p}(t)\Delta t\,\|h\|_{G_\Delta(p^*)}^{p^*}\left(\int_0^{\sigma(x)} w(t)\Delta t\right)^{p-1},
\end{aligned}$$

and, thus

$$\begin{aligned}
&\int_0^\infty \left(\frac{1}{\sigma(x)}\int_0^{\sigma(x)}|f(t)|\,\Delta t\right)^p \Delta x \\
\leq\ & \int_0^\infty (\sigma(x))^{-p}\left(\int_0^{\sigma(x)}|g(t)|^p\, w^{1-p}(t)\Delta t\right)\left(\int_0^{\sigma(x)} w(t)\Delta t\right)^{p-1}\Delta x\,\|h\|^p_{G_\Delta(p^*)}.
\end{aligned}$$

Taking in the last estimate $w(t)=t^{-\frac{1}{p}}$, we obtain that (note that $\frac{1}{\sigma(x)}\leq\frac{1}{x}$)

$$\begin{aligned}
\|f\|^p_{Ces^p_\Delta(\mathbb{T})} &\leq \int_0^\infty x^{-p}\left(\int_0^{\sigma(x)}|g(t)|^p\, t^{1-\frac{1}{p}}(t)\Delta t\right)\left(\frac{x^{1-\frac{1}{p}}}{1-\frac{1}{p}}\right)^{p-1}\Delta x\,\|h\|^p_{G_\Delta(p^*)} \\
&= (p^*)^{p-1}\int_0^\infty\left(\int_0^{\sigma(x)}|g(t)|^p\, t^{1-\frac{1}{p}}(t)\Delta t\right)x^{\frac{1}{p}-2}\Delta x\,\|h\|^p_{G_\Delta(p^*)}.
\end{aligned}$$

Using Lemma (2.3), we obtain that

$$\begin{aligned}
\|f\|^p_{Ces_p} &\leq (p^*)^{p-1}\int_0^\infty\left(\int_t^\infty x^{\frac{1}{p}-2}\Delta x\right)|g(t)|^p\, t^{1-\frac{1}{p}}\Delta t\,\|h\|^p_{G_\Delta(p^*)} \\
&\leq (p^*)^p\int_0^\infty |g(t)|^p\,\Delta t\,\|h\|^p_{G_\Delta(p^*)} = (p^*)^p\,\|g\|^p_p\,\|h\|^p_{G_\Delta(p^*)},
\end{aligned}$$

or $\|f\|_{Ces^p_\Delta(\mathbb{T})}\leq p^*\,\|g\|_p\,\|h\|_{G_\Delta(p^*)}$, that is,

$$L^p_\Delta(\mathbb{T})\cdot G_\Delta(p^*)\subset Ces^p_\Delta(\mathbb{T}),$$

and $\|f\|_{Ces^p_\Delta(\mathbb{T})}\leq p^*!f!$ which proved the left hand side of (3.3). The proof is complete. □

As a consequence from Theorem 3.1, we could get the best form of the dynamic Hardy inequality for $1<p<\infty$ due to Řehák [13] with the same constant.

Corollary 3.1. *Let $\mathbb{T}$ be a time scale with $0\in\mathbb{T}$, f are positive rd-continuous functions defined on $[0,\infty)_\mathbb{T}$. If $1<p<\infty$, then*

$$\int_0^\infty\left(\frac{1}{\sigma(t)}\int_0^{\sigma(t)}|f(t)|\,\Delta t\right)^p\Delta x\leq (p^*)^p\int_0^\infty f^p(x)\Delta x.$$

Proof. *By taking $f(x)=h(x)$ and $g(x)=1$, $x>0$, the right-hand side of (3.3) results the required inequality. This completes the proof.* □

The next two special cases cover known factorizations to both Cesàro sequence and Cesàro function spaces for $p>1$.

Remark 3.1. *If we set $\mathbb{T}=\mathbb{R}$ in Theorem 3.1, we get the factorization to unweighted Cesàro function space due to* [2], [3] *and* [6].

Remark 3.2. *If we set $\mathbb{T}=\mathbb{N}$ in Theorem 3.1, we get the factorization to unweighted Cesàro sequence space due to Bennett* [4].

We conclude this section by presenting the corresponding factorization results for the case $p = 1$. For this case, we will denote

$$\|g\|_\infty := ess\sup_{x>0} |g(x)| < \infty,$$

which allows us to define the space $L^\infty_\Delta(\mathbb{T})$ associated with this norm, and

$$!f!_1 = \inf\left\{\|g\|_\infty \|h\|_{L^1_\Delta(\mathbb{T})}\right\},$$

where infimum is taken over all factorizations $f = g\cdot h$ with $g \in L^\infty_\Delta$ and $h \in L^1_\Delta(\mathbb{T})$. By considering

$$k(x) := \int_x^\infty \frac{\Delta t}{t} \leq \alpha,$$

we will denote by c_1 the least constant for which the above inequality is satisfied. Similarly, c_2 is the biggest constant for which the reverse inequality is satisfied. Assume that there is some positive constant λ for which the inequality $\sigma(x) \leq \lambda x$ holds.

Theorem 3.2. *Let $\mathbb{T}$ be a time scale, then the function f belongs to $Ces^1_\Delta(\mathbb{T})$ if and only if it admits a factorization $f = g \cdot h$ with $g \in L^\infty_\Delta(\mathbb{T})$, $h \in L^1_\Delta(\mathbb{T})$ and*

$$\lambda c_2 !f!_1 \leq \|f\|_{Ces^1_\Delta} \leq c_1 !f!_1. \tag{3.4}$$

Proof. For $g \in L^\infty_\Delta$ and $h \in L^1_\Delta(\mathbb{T})$, we will prove that the factorization $f = g\cdot h$ belongs to $Ces^1_\Delta(\mathbb{T})$. Consider the factorization $f = g \cdot h$. Using Hölder's inequality (2.5) we get that

$$\begin{aligned}\int_0^\infty \left(\frac{1}{\sigma(x)}\int_0^{\sigma(x)} |f(t)|\,\Delta t\right)\Delta x &= \int_0^\infty \left(\frac{1}{\sigma(x)}\int_0^{\sigma(x)} |g(t)\cdot h(t)|\,\Delta t\right)\Delta x \\ &\leq \|g\|_\infty \int_0^\infty \left(\frac{1}{\sigma(x)}\int_0^{\sigma(x)} |h(t)|\,\Delta t\right)\Delta x.\end{aligned}$$

Applying Lemma 2.3, we obtain that $f \in Ces^1_\Delta(\mathbb{T})$ and

$$\|f\|_{Ces^1_\Delta(\mathbb{T})} \leq c_1 \inf\left\{\|g\|_\infty \|h\|_{L^1_\Delta(\mathbb{T})}\right\},$$

where infimum is taken over all possible factorizations of f. This completes the first part of the proof.

Conversely, suppose that $f \in Ces^1_\Delta(\mathbb{T})$ and

$$k(x) = \int_x^\infty \frac{\Delta t}{t} > 0, \text{ for all } t > 0.$$

By setting

$$g(x) = (|f(x)|\, k(x)) \ \operatorname{sgn}\ f(x),\ h(x) = k(x)^{-1},$$

this leads directly to

$$\|g\|_{L^1_\Delta(\mathbb{T})} \leq \lambda \|f\|_{Ces^1_\Delta(\mathbb{T})} < \infty.$$

Moreover, we have that

$$\|f\|_{Ces^1_\Delta} \geq \frac{1}{\lambda} \|h\|_{L^1_\Delta(\mathbb{T})} \geq c_2 \|h\|_{L^1_\Delta(\mathbb{T})} \|g\|_\infty ,$$

which asserts the left-hand side inequality (3.4). This completes the proof. □

4. Factorization theorems of Copson space

Following the same spirit as the previous section, in this section we present the factorization theorems for the Copson spaces Cop^p_Δ, $1 \leq p < \infty$, consisting of all functions f defined on $[0,\infty)_\mathbb{T}$ associated with the norm

$$\|f\|_{Cop^p_\Delta} = \left(\int_0^\infty \left(\int_x^\infty \frac{|f(t)|}{t}\Delta t\right)^p \Delta x\right)^{\frac{1}{p}} < \infty.$$

We define the dynamic function spaces $G_\Delta(p)$ as

$$G^*_\Delta(p) := \left\{ f : \sup_{t>0} \left(\frac{1}{\int_t^\infty x^{-p}\Delta x} \int_t^\infty \frac{f(x)^p}{x^p}\Delta x \right)^{\frac{1}{p}} < \infty \right\},$$

and

$$!!f!!_p = \inf \left\{ \|g\|_p \|h\|_{G^*_\Delta(p^*)} \right\},$$

where infimum is taken over all factorizations $f = g \cdot h$ with $g \in L^p_\Delta(\mathbb{T})$ and $h \in G^*_\Delta(p^*)$.

Theorem 4.1. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. If $1 < p < \infty$, then*

$$Cop^p_\Delta = L^p_\Delta(\mathbb{T}) \cdot G^*_\Delta(p^*), \tag{4.1}$$

which means that the function f belongs to Cop^p_Δ if and only if it admits a factorization $f = g \cdot h$ with $g \in L^p_\Delta(\mathbb{T})$, $h \in G^*_\Delta(p^*)$ and

$$!!f!!_p \leq \|f\|_{Cop^p_\Delta} \leq p^{\frac{1}{p}} (p^*)^{\frac{1}{p^*}} !!f!!_p. \tag{4.2}$$

Proof. "Imbedding ↪". For $f \in Cop^p_\Delta$, $f \neq 0$ and $x > 0$, let

$$k(x) = \frac{1}{x}\int_0^{\sigma(x)} \left(\int_t^\infty \frac{f(s)}{s}\Delta s\right)^{p-1} \Delta t.$$

Consider the factorization $f = g \cdot h$, where

$$g(x) = (|f(x)|\, k(x))^{\frac{1}{p}} \;\operatorname{sgn} f(x) \qquad \text{and} \qquad h(x) = |f(x)|^{\frac{1}{p^*}} k(x)^{\frac{-1}{p}}.$$

As in Theorem 3.1 and applying Lemma 2.3, we can obtain that

$$\|g\|_p^p = \|f\|^p_{Cop^p_\Delta} < \infty,$$

but using Hölder's inequality (2.5) and the definition of $h(x)$, we get that

$$\left(\int_x^\infty \left(\frac{h(t)}{t}\right)^{p^*} \Delta t\right)^p \leq \left(\int_x^\infty \frac{|f(t)|}{t}\Delta t\right)^{p-1} \int_x^\infty \frac{|f(t)|}{t}\frac{k^{-p^*}(t)}{t^{p^*}}\Delta t.$$

We estimate first the right-hand side term of the above inequality multiplied by $\sigma(x)$, we have that

$$\begin{aligned}&\left(\int_0^{\sigma(x)}\Delta t\right)\left(\int_x^\infty\frac{|f(t)|}{t}\Delta t\right)^{p-1}\int_x^\infty\frac{|f(t)|}{t}\frac{k^{-p^*}(t)}{t^{p^*}}\Delta t\\ \leq\ &\left(\int_0^{\sigma(x)}\left(\int_t^\infty\frac{|f(s)|}{s}\Delta s\right)^{p-1}\Delta t\right)\left(\int_x^\infty\frac{|f(t)|}{t}\frac{k^{-p^*}(t)}{t^{p^*}}\Delta t\right)\\ =\ &x\,k(x)\left(\int_x^\infty\frac{|f(t)|}{t}\frac{k^{-p^*}(t)}{t^{p^*}}\Delta t\right),\end{aligned}$$

since, by definition, $\sigma(x)k(x)$ is an increasing function. This implies that

$$\left(\int_0^{\sigma(x)}\Delta t\right)\left(\int_x^\infty\frac{|f(t)|}{t}\Delta t\right)^{p-1}\int_x^\infty\frac{|f(t)|}{t}\frac{k^{-p^*}(t)}{t^{p^*}}\Delta t\leq\int_x^\infty f(t)\frac{k^{1-p^*}(t)}{t^{p^*}}\Delta t.$$

From the definition of $h(x)$, we can write that $h^{p^*}(x)=f(x)k^{1-p^*}(x)$, which leads to

$$\begin{aligned}\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{-1}{p^*}}\left(\int_x^\infty\frac{h^{p^*}(t)}{t^{p^*}}\Delta t\right)^{\frac{1}{p^*}}\ &\leq\ \frac{1}{\left(\int_0^{\sigma(x)}\Delta t\right)^{\frac{1}{p}}\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}}\\ &\leq\ \frac{1}{(p^*-1)^{\frac{1}{p^*}}},\end{aligned}$$

equivalently,

$$\sup_{x>0}\left(\frac{1}{\int_x^\infty t^{-p^*}\Delta t}\int_x^\infty\frac{h^{p^*}(t)}{t^{p^*}}\Delta t\right)^{\frac{1}{p^*}}\leq 1,$$

or $\|h\|_{G_\Delta^*(p^*)}\leq 1$, which means that $h\in G_\Delta^*(p^*)$ and then

$$Cop_\Delta^p\subset L_\Delta^p(\mathbb{T})\cdot G_\Delta^*(p^*),$$

with

$$!!f!!\leq\|g\|_p^p=\|f\|_{Cop_\Delta^p}.$$

"Imbedding $\hookleftarrow$". Let $f=g\cdot h$ with $g\in L_\Delta^p(\mathbb{T})$, $h\in G_\Delta^*(p^*)$ and $w(t)$ be any increasing function on $(0,\infty)_{\mathbb{T}}$, we get by Hölder's inequality (2.5) that

$$\begin{aligned}\int_x^\infty\frac{f(t)}{t}\Delta t\ &=\ \int_x^\infty\frac{g(t)h(t)}{t}\Delta t=\int_x^\infty g(t)w^{-1}(t)\frac{h(t)w(t)}{t}\Delta t\\ &\leq\ \left(\int_x^\infty g^p(t)w^{-p}(t)\Delta t\right)^{\frac{1}{p}}\left(\int_x^\infty\frac{h^{p^*}(t)w^{p^*}(t)}{t^{p^*}}\Delta t\right)^{\frac{1}{p^*}}.\end{aligned}$$

Applying Lemma 2.5 to the term $\left(\int_x^\infty\frac{h^{p^*}(t)w^{p^*}(t)}{t^{p^*}}\Delta t\right)^{\frac{1}{p^*}}$, we get that

$$\int_x^\infty\frac{f(t)}{t}\Delta t\leq\left(\int_x^\infty g^p(t)w^{-p}(t)\Delta t\right)^{\frac{1}{p}}\left(\int_x^\infty\frac{w^{p^*}(t)}{t^{p^*}}\Delta t\right)^{\frac{1}{p^*}}\|h\|_{G_\Delta^*(p^*)},$$

and thus after raising to p^{th} power and integrating from 0 to ∞, we have that

$$\begin{aligned}&\int_0^\infty\left(\int_x^\infty\frac{f(t)}{t}\Delta t\right)^p\Delta x\\ \leq\ & \|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty\left(\int_x^\infty g^p(t)w^{-p}(t)\Delta t\right)\left(\int_x^\infty\frac{w^{p^*}(t)}{t^{p^*}}\Delta t\right)^{p-1}\Delta x.\end{aligned}$$

Using Lemma 2.3, we obtain that

$$\begin{aligned}&\int_0^\infty\left(\int_x^\infty\frac{f(t)}{t}\Delta t\right)^p\Delta x\\ \leq\ & \|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_0^{\sigma(x)}\left(\int_t^\infty\frac{w^{p^*}(s)}{s^{p^*}}\Delta s\right)^{p-1}\Delta t\right)w^{-p}(x)\Delta x.\end{aligned}$$

Taking $w(t)=\left(\int_t^\infty x^{-p^*}\Delta x\right)^{\frac{-1}{pp^*}}$ and applying Lemma 2.2, we can write that

$$\int_t^\infty\frac{w^{p^*}(s)}{s^{p^*}}\Delta s=\int_t^\infty\frac{1}{s^{p^*}}\left(\int_t^\infty\frac{1}{x^{p^*}}\Delta x\right)^{\frac{1}{p^*}-1}\Delta s\leq p^*\left(\int_t^\infty s^{-p^*}\Delta s\right)^{\frac{1}{p^*}}.$$

Inserting this in the above inequality,

$$\begin{aligned}&\int_0^\infty\left(\int_x^\infty\frac{f(t)}{t}\Delta t\right)^p\Delta x\\ \leq\ & (p^*)^{p-1}\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_0^{\sigma(x)}\left(\int_t^\infty\frac{1}{s^{p^*}}\Delta s\right)^{\frac{p-1}{p^*}}\Delta t\right)\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}\Delta x\\ =\ & (p^*)^{p-1}\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}\\ & \times\left(\int_0^{\sigma(x)}\left(\int_0^{\sigma(t)}\Delta s\right)^{\frac{1}{p}-1}\left(\int_t^\infty\frac{1}{s^{p^*}}\Delta s\right)^{\frac{p-1}{p^*}}\left(\int_0^{\sigma(t)}\Delta s\right)^{\frac{p-1}{p}}\Delta t\right)\Delta x.\end{aligned}$$

By the definition of the constant C, we can write

$$\begin{aligned}&\int_0^\infty\left(\int_x^\infty\frac{f(t)}{t}\Delta t\right)^p\Delta x\leq(p^*)^{p-1}\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}\\ & \times\left(\int_0^{\sigma(x)}\left(\int_0^{\sigma(t)}\Delta s\right)^{\frac{1}{p}-1}\left(\frac{1}{(p^*-1)^{\frac{1}{p^*}}}\left(\frac{\sigma(t)}{t}\right)^{\frac{1}{p}}\right)^{p-1}\Delta t\right)\Delta x\\ =\ & (p^*)^{p-1}\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}\\ & \times\left(\int_0^{\sigma(x)}(\sigma(t))^{\frac{1}{p}-1}\frac{1}{(p^*-1)^{\frac{p-1}{p^*}}}\left(\frac{\sigma(t)}{t}\right)^{\frac{p-1}{p}}\Delta t\right)\Delta x\\ =\ & (p^*)^{p-1}\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\left(\int_x^\infty t^{-p^*}\Delta t\right)^{\frac{1}{p^*}}\left(\frac{1}{(p^*-1)^{\frac{p-1}{p^*}}}\int_0^{\sigma(x)}t^{\frac{-1}{p^*}}\Delta t\right)\Delta x,\end{aligned}$$

Finally, after using time scales chain rule (2.4) to estimate the inner integrals, we can easily obtain that

$$\|f\|^p_{Cop^p_\Delta} \le p\,(p^*)^{p-1}\,\|h\|^p_{G^*_\Delta(p^*)}\int_0^\infty g^p(x)\Delta x,$$

or $\|f\|_{Cop^p_\Delta} \le p^{\frac{1}{p}}\,(p^*)^{\frac{1}{p^*}}\,\|g\|_p\,\|h\|_{G^*_\Delta(p^*)}$, that is,

$$L^p_\Delta(\mathbb{T})\cdot G^*_\Delta(p^*) \subset Cop^p_\Delta,$$

and $\|f\|_{Cop^p_\Delta} \le p^{\frac{1}{p}}\,(p^*)^{\frac{1}{p^*}}\,!!f!!$. This completes the proof. □

As a consequence from Theorem 3.1, we could get the best form of the dynamic Copson inequality for $1 < p < \infty$ (see [1]).

Corollary 4.1. *Let $\mathbb{T}$ be a time scale with $0 \in \mathbb{T}$, f are positive rd-continuous functions defined on $[0,\infty)_\mathbb{T}$. If $1 < p < \infty$, then*

$$\int_0^\infty\left(\int_x^\infty \frac{|f(t)|}{t}\Delta t\right)^p\Delta x \le p^{\frac{1}{p}}\,(p^*)^{\frac{1}{p^*}}\int_0^\infty f^p(x)\Delta x.$$

Proof. By taking $f(x) = h(x)$ and $g(x) = 1$, $x > 0$, the right-hand side of (4.2) results the required inequality. This completes the proof. □

The next two special cases cover known factorizations to both unweighted Copson sequence and Copson function spaces for $p > 1$.

Remark 4.1. *If we set $\mathbb{T} = \mathbb{R}$ in Theorem 4.1, we get the factorization to Copson function space due to* [2] *and* [3].

Remark 4.2. *If we set $\mathbb{T} = \mathbb{N}$ in Theorem 4.1, we get the factorization to Copson sequence space due to Bennett* [4, Theorem 5.5].

We conclude this section by presenting the corresponding factorization results for the case $p = 1$. For this case, by denoting $\|g\|_\infty := \sup_{x>0}|g(x)| < \infty$, we get that

$$\int_t^\infty \frac{g^p(x)}{x^p}\Delta x \le \|g\|_\infty\int_t^\infty \frac{1}{x^p}\Delta x,$$

and $!!f!!_1 = \inf\left\{\|g\|_\infty\,\|h\|_{L^1_\Delta(\mathbb{T})}\right\}$, where infimum is taken over all factorizations $f = g\cdot h$ with $g \in L^\infty_\Delta(\mathbb{T})$ and $h \in L^1_\Delta(\mathbb{T})$. Suppose that there is some constant C for which the inequality $\sigma(x) \le Cx$, and assume that there exists a positive constant c_4 the least constant for which this inequality is satisfied. Similarly, c_5 is the biggest constant for which the reverse inequality is satisfied. The proof of the following theorem is similar to the proof of Theorem 3.2 and hence is omitted.

Theorem 4.2. *Let $\mathbb{T}$ be a time scale with $a \in \mathbb{T}$. The function f belongs to Cop^1_Δ if and only if it admits a factorization $f = g\cdot h$ with $g \in L^\infty_\Delta(\mathbb{T})$, $h \in L^1_\Delta(\mathbb{T})$ and*

$$c_5!!f!!_1 \le \|f\|_{Cop^1_\Delta} \le c_4!!f!!_1. \tag{4.3}$$

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